In this Blog series I have collected my comments on Professor Steven Strogatzs's *OPINIONATOR* column on Mathematics published weekly (on Mondays) in The New York Times.

MAIMING THE MIND
Blog #0

In his article *Finding Your Roots* in the OPINIONATOR column of *The New York Times*, March 7, 2010, Professor Steven Strogatz suggests that the equation $x^2 = -1$ has caused an upheaval in mathematics. He goes on to say that such upheavals have been a regular occurrence.

Indeed. Altogether there were four such upheavals during the evolution of the number concept. Each of them arose because of obstruction to the four inverse operations: subtraction, division, logarithm, and root extraction.

In subtracting, the sum and one summand are given, and we seek to find the other summand. It may or may not exist in **N**, the simplest number system consisting of the natural numbers. The obstruction to subtraction is removed by extending the number system. A tentative solution-set is constructed from the natural numbers by pairing the minuends and subtrahends. The trouble is that there are too many candidates for the difference. To ensure uniqueness we introduce a relation by calling two pairs equivalent if the first minuend and the second subtrahend yields the same sum as the second minuend and first subtrahend. The set of equivalence classes is **Z**, the number system that contains the positive as well as the negative integers and zero. In it subtraction can be performed without obstruction, and difference is uniquely determined.

In this extended number system we run into obstruction again on account of division, the inverse operation of multiplication. In dividing, the product and the multiplicand are given, and we seek to find the multiplier. It may or may not exist in **Z**. The obstruction to division is removed by a further extension of the number system. A tentative solution-set is constructed from the integers by pairing the dividend and the divisor (the latter must always be different from zero). Again, the trouble is that there are too many candidates for the quotient. To ensure uniqueness we introduce a relation by calling two pairs equivalent if the first dividend and the second divisor yields the same product as the second dividend and the first divisor. The set of equivalence classes is **Q**, the number system that contains the integers as well as the fractions. In it division can be performed without obstruction, and the quotient is uniquely determined.

We have seen that, although the two direct operations addition and multiplication could always be carried out, the inverse operations of subtraction and division ran into obstructions making it necessary to pass to an extension of the number system. In this way we were led to **Q**, the system of rational numbers wherein the four rules of arithmetic could be performed without obstruction.

Beyond these four rules there are other operations calling for further extensions of the number system. The direct operation of raising a number to some power has *two* inverse operations, logarithm and root extraction. (Two, because this direct operation, unlike addition and multiplication, fails to be commutative: $2^3 = 8 \neq 9 = 3^2$). We run into obstructions in each case. In finding logarithms, given a positive base and a positive power, we seek to find the exponent. Such an exponent may or may not exist in **Q.** The obstruction can be removed in the same way as in previous cases. From rational numbers we construct a tentative solution-set (for details, see the following blog FROM THE RATIONAL TO THE IRRATIONAL). The trouble again is the presence of too many candidates. To ensure uniqueness we introduce a relation by calling two candidates equivalent if they both satisfy the equation. The set of equivalence classes is **R**, the number system consisting of both rational and irrational numbers collectively known as real numbers. Within **R** the logarithm operation can be performed (provided that the power and the base are positive and the latter is different from 1).

The method followed in each case is a quotient set construction. The set of equivalence classes is called a quotient set; it is conceived as a way of „simplifying” a given set. It is akin to the simplification of a fraction via dividing out a common factor of the numerator and the denominator — explaining how quotient sets earn their name („quotient” is Latin for the result of a division).

In mathematics folklore quotient set construction is also used to introduce complex numbers, thus making the development of the number concept a uniform procedure. The other inverse operation of raising a number to the power of another, root extraction, also runs into obstruction, as the notorious example of $\sqrt{-1}$ shows. In **R** there is no number the square of which is –1. A tentative solution-set is offered by geometry. In the set of transformations of plane the quadratic equation $x^2 = -1$ *can* be solved. There are two solutions $\pm x$, the rotations through $\pm 90°$; –1 is rotation through 180° or, what is the same, reflection in a point (explaining why it is not necessary to distinguish between two rotations through $\pm 180°$). The trouble again is that there are infinitely candidates. To see this, let us discard the Euclidean structure of the plane, and replace it with another. The plane has infinitely many Euclidean structures, and every one of them comes with its own rotation i through 90°. While they are different from x, they all satisfy $i^2 = -1$. So, once more, to ensure uniqueness we introduce an equivalence relation for the transformations of the plane. Two transformations are equivalent if they both satisfy the same equation. The set of equivalence classes, in other words, the quotient set of equivalent transformations of the plane is **C**, the system of complex numbers. In it, not only can root extraction be performed without obstruction, but the Fundamental Theorem of Algebra holds stating that every algebraic equation with complex coefficients has at least one root. Counted with multiplicities the number of roots agrees with the degree of the equation.

The evolution of the number concept through the phases **N, Z, Q, R, C** is hereby described through a uniform method: the construction of quotient sets. The picture that comes along is easy and lovely. A complex number no longer appears to be the odd man out. The mystery is taken out of “imaginary numbers” by pointing to the organic relation between algebra and Euclidean geometry.

Nevertheless, the “office of inquisition” does not allow a textbook into the classroom which uses this method of introducing complex numbers — presumably because of the prevailing anti-Euclid ethos of our age. The teacher is forced to toe the line, or out he goes. The student is forced to go through the rigmarole of introducing complex numbers in a wholly artificial and unilluminating way, making them appear to have arrived from the Mars. As they have an “imaginary” component, they can never be fully integrated in the community of “real” numbers. Students are forced to learn new paradigms for the arithmetic operations as applicable to complex numbers, instead of being told to do something they already know how to do: add and multiply them as if they were polynomials in the variable i, and in the end simplify by making the substitution $i^2 + 1 = 0$.

Lest the reader thought it was too far-fetched to suggest that the “office of inquisition” could remove a teacher from the classroom for the offence of putting an unauthorized textbook into the hands of students, let it be stated here that the author of this blog suffered that fate. As a tenured professor at a Canadian university he was defrocked midstream, in full view of the students, for no better reason then refusing to make his students buy the prescribed textbook. Tenure notwithstanding, he was intimidated and threatened to be fired. The case took five years to resolve and the university teachers’ association had to be called in to mediate. The university never apologized.

The way complex numbers are being taught is a scandal and a blot on our “enlightened” educational system. In 1949 the new Chinese government in one of its first decrees outlawed the ancient practice of maiming and deforming the feet of little girls — an age-old cruel custom. But the maiming and deforming of the minds of talented young boys and girls via a thoroughly outmoded way of teaching complex numbers still continues in China, as it is in the rest of the world!