1. Rainbow digits

In this Chapter we solve the problem of constructing an inventory of infinitely many digits. We shall call them rainbow digits, as one of their defining properties is that they follow the colors in the rainbow from red to violet. The other defining property of the rainbow digits is their deep reliance on the decimal number system in general, and the decimal digits in particular.

To be sure, there are number systems that call for such an infinite inventory, for example, Cantor's (see Chapter 3). But the main application of the rainbow digits will probably be in the stepnumber system, which is eminently suitable for working with extremely large numbers. As we shall see, writing a sufficiently large number in stepnumber form takes fewer digits than it does in any other number system. This property can be expressed by saying that the stepnumber system enables one to write very large numbers in their most compact form. There is a reason why Cantor's number system has never made an impact and remained essentially a curiosity. The reason is that it is using the available digits uneconomically, engaging them 'too early', even before lower-ranking digits could have done their work. By contrast, the stepnumber system is most economical with the use of digits: it expresses a number *n*, sufficiently large, in terms of the shortest string of digits among number systems of base *k*, however large *k* may be. Consider also that Cantor's number system puts a new digit to a heavy use immediately after its introduction. By contrast, the stepnumber system uses new digits most sparingly. In this way the usefulness of stepnumbers is greatly enhanced. New digits are only used when absolutely necessary.

The rainbow digits are based on the decimal digits which, when printed in black bold face type, serve at the same time the first ten rainbow digits. They are followed by the same ten digits printed in red bold face type. Thus the first twenty rainbow digits, listed in their natural order, are: **01234567890123456789**. In other words, **0** is the 10^{th}**,** **1** the 11^{th}, **2** the 12^{th}, … , **9** the 19^{th} rainbow digit (**0** is the 0^{th}). The red color is followed by the brown, thus **0** is the 20^{th}, **1** the 21^{st}, **2** the 22^{nd} rainbow digit, etc.

The continuing reliance on the decimal digits has the great advantage of making rainbow digits mnemotechnically superior to any other inventory of infinitely many digits. The ten decimal digits are so deeply rooted in the human psyche that any deviation from them, however clever, would be a detraction. Here, then, are the first one hundred rainbow digits listed in their natural order:

0123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456789

The inventory of infinitely many digits that we are about to construct involves the ten decimal digits colored with the ten standard colors and their shades, with reference to the visible colors as they occur in the spectrum of the rainbow from infra-red to ultra-violet. The frequencies of the ten standard colors, in tHz units (1 tHz = 1 teraHertz = 10^{12}Hz = 10^{6}megaHz; 1Hz = 1/second) are as follows:

FREQUENCIES OF THE TEN STANDARD COLORS (in tHz units)

infrared	red	brown	orange	yellow	green	viridian	blue	indigo	violet
■	■	■	■	■	■	■	■	■	■
400	425	450	475	500	525	550	575	600	625

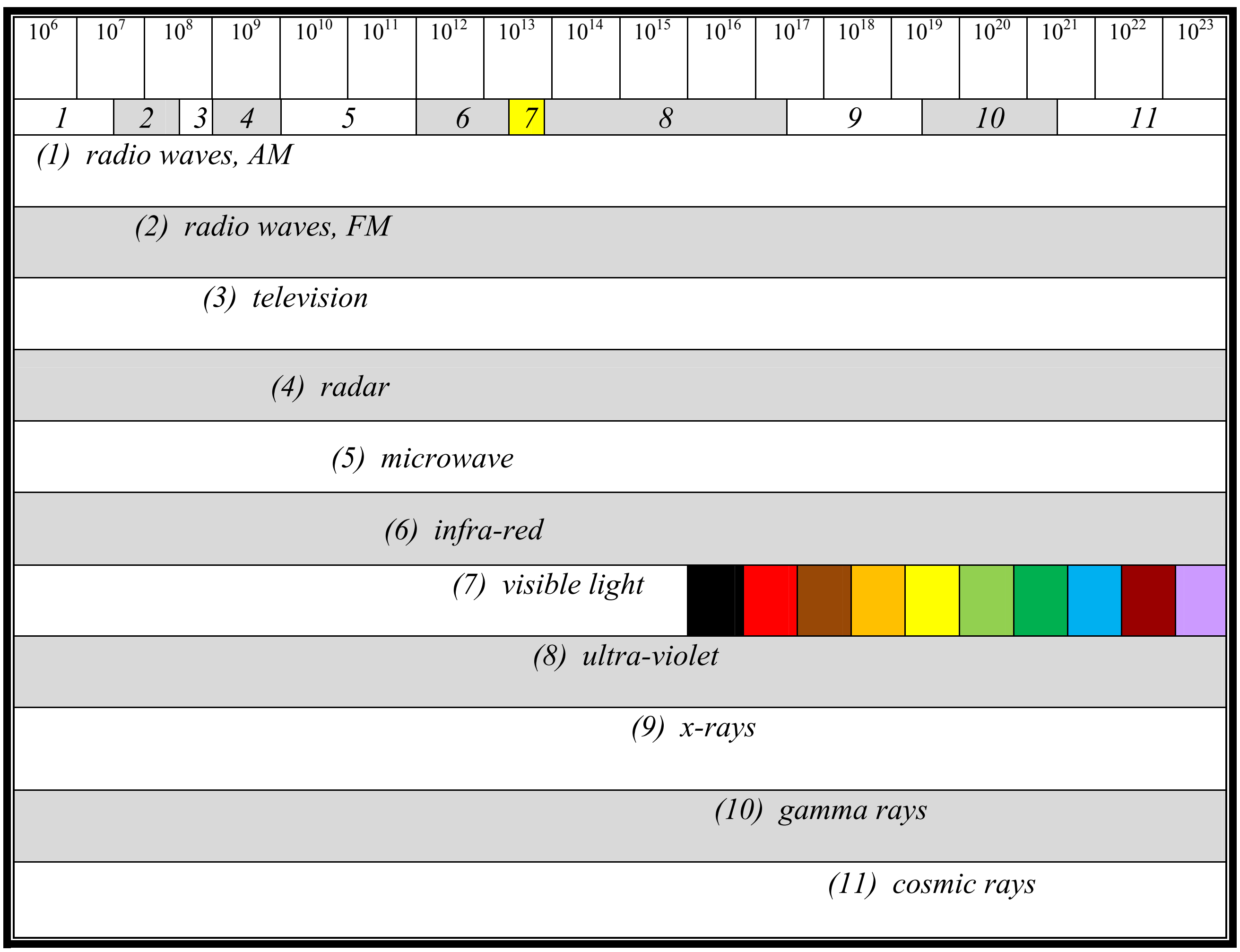
They are pure spectral monochromatic colors. With the exception of black they can be produced by visible light of a single frequency only. The frequency of black we have arbitrarily taken to be that of the invisible infra-red light, 400 tHz.

The visible spectrum of colors is in the range from 425 tHz through 750 tHz. Of this we shall use the range from 425 through 650 as various shades of the standard visible colors, arbitrarily assigning the shades of black to the frequency range 400-425 of invisible infra-red light, as follows:

FREQUENCY RANGES OF THE STANDARD COLORS (in tHz units)

infrared	**red**	**brown**	**orange**	**yellow**	**green**	**viridian**	**blue**	**indigo**	**violet**
400-425	425-450	450-475	475-500	500-525	525-550	550-575	575-600	600-625	625-650

As is well-known from physics, the range for the visible light is but a tiny part of the much wider range of the electro-magnetic radiation, from radio frequencies (about 10^6 Hz at the low end) to cosmic rays (about 10^{24} Hz at the high end of the spectrum):



The fact that the spectrum of visible lights is minuscule is a consequence of the choice of the unit of time, 1 sec (the second). With a smaller unit we could zoom in. Herewith we change the unit of time. We shall call the new unit "tooth" (plural: "teeth"), abbreviated "tth". The conversion formulas are: 1 tth $= 4(10^{-13})$sec; 1 sec $= 25(10^{11})$tth. With this change the frequencies of the standard colors become the integers 16 (for the color black), 17 (for the color red), through 25 (for the color violet). This reveals the reason for the choice of the new unit, which is mnemotechnics. The frequencies of the standard colors are easily remembered if we consider that 16 + 9 = 25 or, what is the same, $3^2 + 4^2 = 5^2$ where 3, 4, 5 are the smallest non-trivial Pythagorean numbers. We have:

FREQUENCIES AND FREQUENCY RANGES OF THE STANDARD COLORS (in tth units)

infrared	**red**	**brown**	**orange**	**yellow**	**green**	**viridian**	**blue**	**indigo**	**violet**
16	17	18	19	20	21	22	23	24	25
16-17	17-18	18-19	19-20	20-21	21-22	22-23	23-24	24-25	25-26

For the sake of comparison, here are the lower limits of some other frequency ranges of the electro-magnetic spectrum in terms of tth units:

micro	*infrared*	*light*	*UV*	*x-ray*	*gamma*
$12(10^{-4})$	$12(10^{-2})$	17	30	$12(10^{4})$	$12(10^{6})$

According to physics, every real number between 16 and 26 is the frequency of a unique monochromatic color. These colors are available to choose from in coloring rainbow digits with two modifications:

(i) at the lower end of the spectrum, the frequencies from 16 to 17 include the invisible infra-red light that we use as proxy for the standard color black (frequency 16) and its shades of grey.

(ii) at the upper end of the spectrum, the higher range of the color violet from frequency 26 through 30 is excluded from consideration as colors for the rainbow digits for being redundant.

The set of colors with frequency from 16 through 26 is not countable and is, therefore, too big for the purpose of coloring the rainbow digits. We need only a denumerable subset. Fortunately, there is an obvious choice: the subset of monochromatic colors whose frequencies are *rational* numbers. Even though this means that we disregard the vast bulk of monochromatic colors, those whose frequencies are *irrational* numbers, no harm done. The eye will not be able to notice the loss. Nor can it distinguish between colors whose frequencies are rational *versus* irrational numbers. After all, this distinction depends on the choice of the unit of time.

Thus we have a one-to-one correspondence between the colors of the rainbow digits and the rational numbers between 16 and 26, namely, the latter are the frequencies of the former in tth units. Remember that for each color as specified by this one-to-one correspondence there are ten rainbow digits corresponding to the ten decimal digits.

When we deal with the rational numbers between 16 and 26, the old-fashioned mathematical terms of "characteristic" and "mantissa" come handy. The *characteristic* of a positive rational number is its integral part: the digits preceding the decimal point; its *mantissa* is the fractional part: the digits following the decimal point.

The *grade* of a color is the number of digits of the mantissa of its frequency. If the mantissa is zero, then $k = 0$ and we have one of the standard colors. If the frequency is an infinite repeating decimal (more precisely, if the rational number cannot be written as a finite decimal), then $k = \infty$. The number of first grade colors is one hundred (ten for every standard color); there are one thousand second grade colors (ten for every first-grade color); there are ten thousand third grade colors (ten for every second-grade color). In general, the number of k^{th} grade colors is 10^{k+1} (ten for every $(10^{k-1})^{\text{st}}$-grade color). It is left as an exercise for the reader to discuss the case $k = \infty$.

Even though the naked eye may not be able to distinguish between two shades of a color of the same grade, optical instruments and the computer can. Remember, the problem of telling apart shades of two rainbow digits is a theoretical problem, not a practical one. As we have already pointed out, the colors black and red are all we need for writing stepnumbers that occur in practice. Recalling the pertinent facts from physics is sufficient to complete the proof of the existence of an inventory of infinitely many step-digits.

As we can enumerate the rational numbers, so we can enumerate the colors of the rainbow digits. We have infinitely many shades for each of the ten standard colors, and they are sorted out by the mantissas of their frequencies. Enumerating the consecutive colors of the rainbow digits is done through the enumeration of their frequencies grade by grade, thus:

16, 17, 18, 19, 20, 21, 22, 23, 24, 25,
16.1, 17.1, 18.1,…, 25.1, 16,2, 17.2, 18.2,…, 25.2,16.3,…, 16.9, 17.9, 18.9,…,25.9,
16,01, 17.01,…, 25.01,16.02, 17.02,…,25.02, 16.03,…, 25,98, 16.99, 17.99,…,25.99,
16.001,…, 25.001, 16.002,…, 25.002, 16.003,…, 25.998, 16.999,…, 25.999,…

Note that the enumeration preserves the natural order of rational numbers, as well as the natural order of colors in the rainbow. A break in the natural order only occurs when we reach the top frequency for the color violet, grade k, namely, 25.99…9 (k digits of 9's), for $k = 0, 1, 2,\ldots$ At these points the frequency drops back to the range 16-17 of shades for the color black. In other words, the frequency 25.99…9 (k digits of 9's) is followed by the frequency 16,00…01 (k digits of 0's) as the starting frequency for the shade of black, grade $k + 1$.

Note that in the sequence of the top frequencies for the color violet, 25.9, 25.99, 25.999,… the general term, 25.99…9 (k digits of 9) tends to the frequency 26 as $k \to \infty$. At the other end of the spectrum, in the sequence of the starting frequencies for the color black, 16.1, 16.01, 16.001,…, the general term 16.00…01 (k digits of 0) tends to the frequency 16 as $k \to \infty$.

The typographical rendering of shades of grade two

Again, the following problem has only theoretical interest because in practice the only two colors that are likely to occur are black and red. We have already seen the list of the first one hundred rainbow digits (e.g.,see the frieze on the title page of the book). We now want to have the list of the first one thousand rainbow digits. First we have to solve the subjective problem that the naked eye may not be able to tell apart different shades of the same grade of the same standard color. For this reason we have to use the typographical device of modifying the font by using a combination of italic, bold, strikeout, double strikeout face types in order to find substitutes for the different shades. The ten suggested combinations of face types are as follows:

1234 1234 1234 12341234 1234 1234 1234 1234 1234 (bold)

1234 1234 1234 12341234 1234 1234 1234 1234 1234	(italics+bold)
~~1234 1234 1234 12341234 1234 1234 1234 1234 1234~~	(strikeout)
~~1234 1234 12341234 1234 1234 1234 1234 1234 1234~~	(strikeout+italics)
~~1234 1234 1234 1234 1234 1234 1234 1234 1234 1234~~	(double strikeout)
~~1234 1234 12341234 1234 1234 1234 1234 1234 1234~~	(double strikeout+italics)
1234 1234 1234 12341234 1234 1234 1234 1234 1234	(normal)
1234 1234 1234 1234 1234 1234 1234 1234 1234 1234	(italics)
1234 1234 1234 12341234 1234 1234 1234 1234 1234	(outline+bold)
1234 1234 12341234 1234 1234 1234 1234 1234 1234	(outline+bold+italics)

The first one thousand rainbow digits

0123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456789
0123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456789
~~0123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456789~~
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Read as a single stepnumber, it is the largest one the expansion of which consists of 999 digits. The stepnumber expansion of the next number, b_{1000}, consists of 1000 stepdigits. By contrast, the decimal expansion of b_{1000} consists of 1928 decimal digits, almost twice as many!

Recall the notation $\boldsymbol{n! = 123...n}$ and the formula $\boldsymbol{n! + 1 = 10_n}$. Thus we have:
1! = 1, 2! = 12, 3! = 123,…, **9! = 123456789**; e.g., **9! + 1 = 123456789 + 1 = 1000000000 = 10_9** = b_{10}. We extend this notation to the rainbow digits:

0! = 1234567890   **0! = 12345678901234567890**
1! = 12345678901   **1! = 123456789012345678901**
2! = 123456789012, etc.   **2! = 1234567890123456789012**, etc.

Then we have:

9! + 1 = b_{20} = **10_{19}**   **9! + 1** = b_{30} = **10_{29}**   **9! + 1** = b_{40} = **10_{39}**
9! + 1 = b_{50} = **10_{49}**   **9! + 1** = b_{60} = **10_{59}**   **9! + 1** = b_{70} = **10_{69}**
9! + 1 = b_{80} = **10_{79}**   **9! + 1** = b_{90} = **10_{89}**   **9! + 1** = b_{100} = **10_{99}**

To recapitulate, the colors of the rainbow digits are precisely those monochromatic colors of the rainbow whose frequencies are rational numbers from 16 through 26 in tth units. They furnish an inventory of denumerably many stepdigits which enable one to write ever larger stepnumbers without taxing the human memory (let alone computer memory that, contrary to myth, can be taxed, too).

The stepnumber system is most economical with the use of digits in two ways. *First*, the expansion of a number, sufficiently large, takes a shorter string of digits in the stepnumber system than it would in any other with base *k*, however large *k* may be. For example, every *n*-digit decimal can be expanded as a stepnumber of *fewer* than *n* digits whenever *n* is at least 57. This follows from the inequality $b_n > 10^n$ valid for all $n \geq 57$. In point of fact, the decimal expansion of b_{57} has 57 digits.

Second, the $(n+1)^{st}$ digit is not pressed into service until the full potential of the first n digits has not been exploited and, even then, it is used as sparingly as possible. Let us demonstrate this economy through the following example.

Having counted to **1234567899** $= b_{11} - 2 = 678{,}568$ we have exhausted the black stepdigits. In order to carry on we must introduce the red rainbow digits, the first of which is **0**. Now we can write **1234567899 + 1 = 1234567890 = 0!** In counting forward from here on we shall not encounter another red digit until we get to **10234567890**, a block $b_{11} - b_{10} = 562{,}595$ long: **10000000000**, **10000000001**, **10000000002, 10000000010, 10000000011,**…, **10234567899**, **10234567890**,… Between the second and third occurrence of **0,** that is, between the stepnumbers **10234567890** and **11234567890,** is another block of the same length. The next block from **11234567890** through **12034567890** is 372,939 long and this length occurs three times.

Lest one think that these are random numbers, we penetrate the matter further. In the following table we list the stepnumbers in which the only colored digit, **0**, occurs exactly once (with the exception of the last, in which it occurs twice).

1234567890	**12341567890**	**12345617890**	**12345675890**	**12345678890**	**12345678900**
10234567890	**12342567890**	**12345627890**	**12345676890**	**12345678900**	**12345678901**
11234567890	**12343567890**	**12345637890**	**12345677890**	**12345678910**	**12345678902**
12034567890	**12344567890**	**12345647890**	**12345678090**	**12345678920**	**12345678903**
12134567890	**12345067890**	**12345657890**	**12345678190**	**12345678930**	**12345678904**
12234567890	**12345167890**	**12345667890**	**12345678290**	**12345678940**	**12345678905**
12304567890	**12345267890**	**12345670890**	**12345678390**	**12345678950**	**12345678906**
12314567890	**12345367890**	**12345671890**	**12345678490**	**12345678960**	**12345678907**
12324567890	**12345467890**	**12345672890**	**12345678590**	**12345678970**	**12345678908**
12334567890	**12345567890**	**12345673890**	**12345678690**	**12345678980**	**12345678909**
12340567890	**12345607890**	**12345674890**	**12345678790**	**12345678990**	**12345678900**

Note that the last 12 entries are consecutive stepnumbers. In the next table we have numbered the 65 blocks consisting of consecutive stepnumbers demarcated by the entries in the previous table. In the adjacent column we have indicated the lengths of blocks:

1.	562,595	12.	73,013	23.	4,516	34.	757	45.	11	56.	1
2.	562,595	13.	73,013	24.	4,516	35.	757	46.	11	57.	1
3.	372,939	14.	73,013	25.	4,516	36.	101	47.	11	58.	1
4.	372,939	15.	20,878	26.	4,516	37.	101	48.	11	59.	1
5.	372,939	16.	20,878	27.	4,516	38.	101	49.	11	60.	1
6.	190,497	17.	20,878	28.	757	39.	101	50.	11	61.	1
7.	190,497	18.	20,878	29.	757	40.	101	51.	11	62.	1
8.	190,497	19.	20,878	30.	757	41.	101	52.	11	63.	1
9.	190,497	20.	20,878	31.	757	42.	101	53.	11	64.	1
10.	73,013	21.	4,516	32.	757	43.	101	54.	11	65.	1
11.	73,013	22.	4,516	33.	757	44.	101	55.	1		

The lengths of blocks as given in this table are just the spectral coefficients in the rows of Pascal's triangle for the spectral coefficients, see Chapter 4.

By way of checking, we add the lengths of all the 65 blocks and we expect to get the length of the big block from the first **0** to **00:**

$$
\begin{array}{rcr}
2(562{,}595) & = & 1{,}125{,}190 \\
+\,3(372{,}939) & = & 1{,}118{,}817 \\
+\,4(190{,}497) & = & 761{,}988 \\
+\,5(73{,}013) & = & 365{,}065 \\
+\,6(20{,}878) & = & 125{,}268 \\
+\,7(4{,}516) & = & 31{,}612 \\
+\,8(757) & = & 6{,}056 \\
+\,9(101) & = & 909 \\
+\,10(11) & = & 110 \\
+\,11(1) & = & 11 \\
\hline
 & = & 3{,}535{,}026
\end{array}
\quad \text{which is 1 less than} \quad
\begin{array}{rr}
b_{12} - b_{11} = & 4{,}213{,}597 \\
- & 678{,}570 \\
\hline
 & 3{,}535{,}027
\end{array}
$$

We see huge gaps between any two occurrences of **0** in the beginning, an indication how sparingly **0** is used. By the end of the table these gaps will become minimal (in the last column the stepnumbers are consecutive.)

As the last table shows, in the sequence of the first 2 million consecutive stepnumbers **0** occurs no more than 4 times! In the sequence of the first 4,213,597 stepnumbers colored digits occur only 66 times, with a heavy concentration in the latter part.

The point of this example is to reveal the pattern followed by the appearance of every new digit as we count forward. The general rule is given by the Blockbuster Formula of Chapter 4. It shows how the block of stepnumbers $\leq$ **(*n* + 1)!** is split by the consecutive occurrences of the digit ***n***.

In that Chapter a similar survey is carried out for the first 15 occurrences of the digit **4** (see Example, following the Overflow Formula on p 44), which can be easily followed on the Table of Consecutive Stepnumbers in the Introduction.

*   *   *

On the following page we list the names of the first one hundred (of the infinitely many) rainbow digits for the stepnumber system. The principle of forming these names was already mentioned in the Introduction. Recall the role of the five vowels **a, e, i, o, u** and their alphabetical order play.

In the Exercises we shall see how the list of the names of digits, as well as the list of the names of milestones, can be extended.

NAMES OF DIGITS

0	ala	0	alya
1	ale	1	alye
2	ali	2	alyi
3	alo	3	alyo
4	alu	4	alyu
5	alla	5	allya
6	alle	6	allye
7	alli	7	allyi
8	allo	8	allyo
9	allu	9	allyu
0	ela	0	elya
1	ele	1	elye
2	eli	2	elyi
3	elo	3	elyo
4	elu	4	elyu
5	ella	5	ellya
6	elle	6	ellye
7	elli	7	ellyi
8	ello	8	ellyo
9	ellu	9	ellyu
0	ila	0	ilya
1	ile	1	ilye
2	ili	2	ilyi
3	ilo	3	ilyo
4	ilu	4	ilyu
5	illa	5	illya
6	ille	6	illye
7	illi	7	illyi
8	illo	8	illyo
9	illu	9	illyu
0	ola	0	olya
1	ole	1	olye
2	oli	2	olyi
3	olo	3	olyo
4	olu	4	olyu
5	olla	5	ollya
6	olle	6	ollye
7	olli	7	ollyi
8	ollo	8	ollyo
9	ollu	9	ollyu
0	ula	0	ulya
1	ule	1	ulye
2	uli	2	ulyi
3	ulo	3	ulyo
5	ulla	5	ullya
6	ulle	6	ullye
7	ulli	7	ullyi
8	ullo	8	ullyo
9	ullu	9	ullyu

Exersises

1. Count forward and then count backward to find the next five stepnumbers in either direction, starting from every one of the 66 stepnumbers in the second last table of this Chapter.

2. Count forward and then backward from the stepnumber **12345678950** and see that the next 100 stepnumbers in either direction are written without recourse to colored digits. Write the 101^{st} stepnumber in either direction.

3. Count forward and then count backward to find the next five stepnumbers in either direction, starting with:

   (*a*) **0!0,** (*b*) 0!0**,** (*c*) **4!4,** (*d*) **4!4,** (*e*) **4!4,** (*d*) **9!9**, (*e*) **9!9,**
   (*f*) **9!9,** (*g*) **4!0$_4$**, (*h*) **4!4$_4$**, (*i*) **4!0**$_4$

4. After the first appearance of the n^{th} digit, how far forward do we have to count in order to get to the second, third, and fourth appearance of that digit?

5. What is the ordinal of the stepnumber **1234567890123456789012345678900** ?

6. The $(b_{11} - 1)^{st}$ stepnumber is **0!** Write the $(b_{31} - 1)^{st}$, $(b_{41} - 1)^{st}$, . . . , $(b_{91} - 1)^{st}$ stepnumbers.

7. What is the ordinal of the stepnumbers **5!**, **5!9!**, **5!5! ?**

8. True or false? Every finite decimal fraction between 17 and 26 is the frequency of a shade of of the standard colors. If the answer is "yes", describe how to find that shade of color.

9. Write a few stepnumbers of each of the following shade of color, if their frequencies are given in tth units as 16.66, 17.76, 17.89, 18.48, 19.14, 19.45, 19.99, 20.01

10. How many stepnumbers can be written without recourse to colored digits between the pair **123456789051** and **123456789061**? What is the answer if the given pair is **123456789015** and **123456789016**?

11. Write the following 100-digit stepnumber
    1234567890123456789012345678901234567890123456789012345678901234567890123456789
    01234567890123456789012345678 99
    in a more compact form. Write the next stepnumber in a more compact form.

12. We have seen that **0** occurs in 66 stepnumbers ≤ 10_{10}. In how many stepnumbers ≤ 10_{11} does **1** occur?

13. ***n*!** is the first stepnumber in which the digit ***n*** occurs. Prove that the number of all stepnumbers $\leq$ **(*n* + 1)!** in which the digit ***n*** occurs is $\binom{n+1}{2} = \frac{n(n+1)}{2}$.

14. Extend the table for consecutive stepnumbers in the Introduction from 300 to 600.

15. Extend the table further from 600 to 900.

16. Based on your extended tables, make a survey on the occurrences of the digit ***n*** among the consecutive stepnumbers $\leq$ **(*n* + 1)!** as the survey carried out above for the digit **0,** for ***n*** = **2**, **3**, **4**, **5**, and **6.**

17. Check that the first occurrences of the digit ***n*** among the consecutive stepnumbers $\leq$ **(*n* + 1)!** conform to the same rule that governs the first occurrences of the digit **0,** for ***n*** = **2**, **3**, **4**, **5**, and **6.**

18. Count up to the binary number $\mathbf{10_5}$, filling in the names for the dots, thus: **ala, ale, mala, mala ale, bala, bala ale, bala mala, bala mala ale, trala, trala ale, trala mala, trala mala ale, trala bala,...**, **pentala.**

19. Count up to the stepnumber $\mathbf{10_4}$, filling in the names for the dots, thus: **ala, ale, mala, ale ale, ale ali, bala, ale ala ale, ale ala ali, ale ale ala, ale ale ale, ale ale ali, ale ali ala, ale ali ale, ale ali ali, ale alia lo, trala, ale ala ala ale, ale ala ala ali, ale ala ale ala,**…, **quadrala.**