SQUARE DANCING ON SQUARE TILES
Blog #3

Professor Steve Strogatz published an article with the title SQUARE DANCING in the OPINIONATOR column of *The New York Times* on March 14, 2010. In this blog I would like to discuss a refinement that waltzes thousands of years back into history to Babylonian times.

Draw a right triangle with sides 3 and 4 units long; then the hypotenuse will be 5 units long. Draw squares on the sides and on the hypotenuse and show how they are made up of 9, 16, and 25 unit squares, resp.

The result can be seen in http://demonstrations.wolfram.com/PythagoreanTriples
We call and denote this as the Pythagorean triple (3, 4, 5), and pose the problem of finding all possible ways of "square dancing on square tiles".

Let's look at the sequence of squares 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225,… and pick two the sum of which is also in the list. We skip (6, 8,10) and (9,12,15) because they can be trivially obtained from (3, 4, 5) by doubling and trebling components. In other words, the components have a common multiple other than 1 and therefore are not coprime. A Pythagorean triple is called primitive if its components are coprime. Here is a genuinely new one that is primitive: 25 + 144 = 169, leading to the Pythagorean triple (5, 12, 13). The next ones would be (8, 15, 17) and (7, 24, 25). If you tried to find more, you would see that square dancing on square tiles never comes to an end.

To find all possible solutions we go to Euclid's formula stating that from

$$(m^2 - n^2,\ 2mn,\ m^2 + n^2)$$

where m, n are coprime positive integers satisfying $m > n$, of which one is even and the other odd, we can get all primitive Pythagorean triples. From the primitive ones, in turn, we can get all of them by multiplying components with an arbitrary positive integer k. The special case $n = 1$ was already known to the Babylonians.

For more, see: www.wikipedia.org/wiki/Pythagorean_triple

In one way or another all Pythagorean triples come from (3, 4, 5). One component of any Pythagorean triple is divisible by one of 3, 4, 5. Possibly the same component is divisible by all three, or by any two of them as in (11, 60, 61), (5, 12, 13), (8, 15, 17), (20, 21, 29). This occurs when one or two of the components are prime numbers. In particular, the product of the two smaller components is always divisible by $3\times4 = 12$, and the product of all three components is always divisible by $3\times4\times5 = 60$. This could be useful as a check.

A beautiful example of contact between geometry and algebra is the theorem stating that all Pythagorean triples can be derived from the simplest one (3, 4, 5), also known as the parent-child relationship (ibid.) I describe it in an unusual way as changing the metric structure of the plane. Discard the Euclidean structure of the plane and replace it with a Lorentz structure (in which the circles are equilateral hyperbolas). Then there are three distinguished Lorentzian transformations of the plane, such that any given Pythagorean triple is obtained by applying one or more of the distinguished transformations to (3, 4, 5) once or several times.

EXERCISES.

(1) Show that there are no Pythagorean triples such that every component is an odd number.

(2) Are there Pythagorean triples such that two components are prime numbers? If so, how to look for them?

(3) Are there Pythagorean triples such that all three components are prime numbers?

(4) Find Pythagorean triples one of whose components is (*a*) 56; (*b*) 35; (*c*) 42.

ANSWERS.

(1) No. By Euclid's formula the middle component must be even.

(2) Yes. We find them by checking components for divisibility by 12, 15, 20, or 60.

(3) No. All prime numbers with the exception of 2 are odd, so by (1) the only possibility is (b, 2, c), that is, $b^2 + 4 = c^2$, or $(c + b)(c - b) = 4$. But 4 can only be factorized as 2×2 or 1×4; either one is impossible as b, c are odd prime numbers.

(4) (*a*) (33, 56, 65), another one: (56,783, 785).
(*b*) (12, 35, 37), another one: (35, 612, 613).
(*c*) There is none. To see this we observe that there are only 16 primitive Pythagorean triples with $c < 100$ as follows:

(3,4,5)	(5,12,13)	(7,24,25)	(8,15,17)
(9,40,41)	(11,60,61)	(12,35,37)	(13,84,85)
(16,63,65)	(20,21,29)	(28,45,53)	(33,56,65)
(36,77,85)	(39,80,89)	(48,55,73)	(65,72,97)

It can be seen that 42 does not occur among the components. If there was a Pythagorean triple with 42 as a component, then it would have to be the snallest component and we would have:
$u^2 - v^2 = (u + v)(u - v) = 42 = 7×6 = 14×3 = 42×1 \Rightarrow u + v = 7, u - v = 6$
or $u + v = 14, u - v = 3$ or $u + v = 42, u - v = 1$. In neither case is there an integral solution for u, v.

(Revised March 20, 2010.)