Chapter 3: ARDHACCHEDA OF VIRASENA (816 A.D.)

By a hair's breadth the Romans missed inventing logarithms. As we have seen in Chapter 2, if you want to multiply any number by a power of two, say 2^n, you just double that number n times. The desired product is the result of the last doubling. You may just forget about the adding part of the Roman method of multiplication. The exponent n actually tells you how many times you have to double the given number. It also tells you how many times you can halve 2^n to get 1.

You can actually extend this result to the general case when the other factor of the product is not a power of two, by introducing the concept of "fractional doubling". At any rate, that is what the Indian mathematician Acharya Virasena did. He considered 2^x for fractional values of x to yield the number which, when halved x times, results in 1. In modern terminology we would say $x = \log_2 2^x$ (read: logarithm with base 2 of 2^x).

In other words, the Romans could have introduced the definition:

$\log_2 n$ = the number of times n must be halved in order to get 1

This definition would have led them to compile the table:

n	1	2	4	8	16	32	64	128	256	512	1024
$\log_2 n$	0	1	2	3	4	5	6	7	8	9	10

With pro-rating of the increases in that table (or, to use the technical term, applying interpolation), they would have extended their table to cover all integers:

n	1	2	3	4	5	6	7	8	9	10
$\log_2 n$	0	1	1.59	2	2.32	2.59	2.81	3	3.17	3.32

The fractional part of the value of $\log_2 n$ indicates that the last halving overshot the target 1, so it had to be reduced to "fractional halving" to put it back on target. In the same way they could have extended their table to fractional values of $n > 1$. Such a table can be found on the Internet, see: *Base 2 Logarithms Table*, http://webcache.googleusercontent.com. The 2-decimal place value, of course, is only an approximation. For a 16-decimal place value, see the website: www.rapidtables.com. It is understandable that the Romans missed their chance to become the inventors of logarithms two thousand years ago. Their number system was just too clumsy. Virasena was enormously helped by the Hindu decimal number system that came into use in India at about the same time, between 800 and 825 A.D. Thus it took eight and one half of a century before the idea of counting the number of halvings in finding a product resurfaced. Acharya Virasena was also an orator and a poet, and a student of the Jain sage Elacharya. He introduced the concept *ardhaccheda*, the number of times n could be *halved* before we reach 1 (or, if $0 < n < 1$, the number of times n could be *doubled* before we reach 1, in which case the ardhaccheda of n is distinguished by giving it the negative signature). As we mentioned already, nowadays we call ardhaccheda "logarithm to base 2". Virasena also worked out *trakacheda* and *caturthacheda* that nowadays would be called logarithms to base 3 and 4, denoted $\log_3 n$ and $\log_4 n$, respectively.

Virasena knew how the tables above could be used to simplify calculations, e.g., multiply two numbers m and n by adding their logarithms and taking the "antilogarithm" of the sum, which means finding $\log_2 m + \log_2 n$ in the second row of the table and pass to the corresponding value of n in the first. For example, in finding 4×8 we write:

$$\log_2(4 \times 8) = \log_2 4 + \log_2 8 = 2 + 3 = 5$$

which we find in the second row just under $n = 32$, confirming that $4 \times 8 = 32$.

"Taking the antilogarithm of $\log_2 n$" is a perfectly superfluous term, as its meaning coincides with that of "taking 2 to the power of $\log_2 n$": $2^{\log_2 n} = n$. The two equalities $5 = \log_2 32$ and $2^5 = 32$ have exactly the same meaning, namely, $x = 5$ is the solution to the equation $2^x = 32$. To distinguish between the two expressions we call the first *logarithmic form*, the second the *exponential form*. The contents of either one is that 2 must be raised to the power of 5 in order to get 32. Moreover, 5 is called the *exponent* or *logarithm*; 2 is the *base*; and 32 is the *power*.

In general, let the base be b ($b > 0$, $b \neq 1$), the exponent x, and the power y ($y > 0$). Then $x = \log_b y$ is the solution of the exponential equation $b^x = y$. In this general case we can also see that logarithm is the exponent to which the base has to be raised in order to get the power. The logarithm satisfies the following identities:

$$b^{\log_b x} = x, \quad \log_b b^x = x$$

$$\log_b(x_1 x_2) = \log_b x_1 + \log_b x_2$$

$$\log_b\left(\frac{1}{x}\right) = -\log_b x \qquad (x \neq 0)$$

$$\log_b x^n = n \log_b x$$

$$\log_b\left(\sqrt[n]{x}\right) = \frac{\log_b x}{n}$$

The special cases $\log_b b = 1$ and $\log_b 1 = 0$ deserve special mention.

Exercises:

1. Write in logarithmic form the following:

(*a*) $2^5 = 32$, (*b*) $5^2 = 25$, (*c*) $10^4 = 10000$, (*d*) $2^{-3} = 0.125$

2. Write in exponential form the following:

(*a*) $\log_{10} 100 = 2$, (*b*) $\log_2 0.0625 = -4$, (*c*) $\log_5 0.0064 = -6$,

(*d*) $\log_5 3125 = 5$, (*e*) $\log_2 2048 = 11$

3. Evaluate and provide reasons: (*a*) $\log_2 128$, (*b*) $\log_5 0.00128$,

(*c*) $\log_{10} 0.0001$, (*d*) $\log_{10} 1000$, (*e*) $\log_2 0.25$, (*f*) $\log_5 0.16$,

(*g*) $\log_b b^n$

4. Evaluate and provide reasons: (*a*) $\log_2 (¼)$, (*b*) $\log_5 (1/25)$,

(*c*) $\log_5 25$, (*d*) $\log_2 512$, (*e*) $\log_{10} 0.01$, (*f*) $\log_b b^n$, (*g*) $\log_b(\sqrt[3]{b})$,

(*h*) $\log_6(\sqrt[3]{6})$, (*i*) $\log_b(x/y)$, (*j*) $\log_b(\sqrt{x})$, (*k*) $\log_b(\sqrt[m]{x^n})$

5. Solve the exponential equations and check:

(*a*) $2^x = 32$, (*b*) $5^x = 0.032$, (*d*) $2^x = 16$, (*e*) $5^x = 0.16$,
(*f*) $10^x = 1/1000$, (*g*) $2^{2x-1} = 1024$

6. Solve the logarithmic equations and check:

(*a*) $\log_{10} 10^x = 1/10$, (*b*) $\log_2 x = 128$; (*c*) $\log_5 x = 0.00128$,

(*d*) $\log_{10} (\log_{10} x) = 1$, (*e*) $\log_2 5^{3x+1} = 2$

7. Fill the blanks in the following tables:

(*a*)

n	1	0.2	0.4	0.8	0.16					
$\log_2 n$										

(*b*)

n	1	0.5	0.25	0.125						
$\log_5 n$										

(*c*)

n	1	10	10^2	10^3						
$\log_{10} n$										

(*d*)

n	1	0.1	0.01	0.001						
$\log_{10} n$										

(*e*)

n	1	5	25							
$\log_5 n$	0	1	2	3	4	5	6	7	8	9

Change of basis

There are *direct* and *inverse* operations. Direct operations are addition, multiplication, and raising one number to the power of another. Each of these have inverse operations. The inverse operation of addition is subtraction: given $x + a = b$, for the unknown x the solution is given by the difference $x = b - a$. The inverse operation of multiplication is division: given $ax = b$ ($a \neq 0$), for the unknown x the solution is given by the fraction $x = b/a$. Addition and multiplication are commutative operations, and therefore there is just one inverse operation for either. This is no longer the case for the third direct operation which is not commutative ($2^3 = 8 \neq 9 = 3^2$) and, therefore, in the equation $b^n = m$ either b or n can be unknown, and the choice gives us *two* indirect operations. If the unknown is b, then the inverse operation yielding it is *root extraction*, $b = \sqrt[n]{m}$. If the unknown is n then, as we know, we have an exponential equation whose solution can be obtained by finding logarithm: $n = \log_b m$. We conclude that *the direct operation of raising one number to the power of another has two inverse operations: root extraction and finding logarithm.*

We have seen that logarithms simplify the operations multiplication, division, raising to powers, and root extraction by reducing them to addition, subtraction, multiplication and division, respectively. We now face the question: is the other inverse operation of the power operation also simplified, and if so, how. The answer to this question is that the other inverse operation is also simplified, namely, it is reduced to division as well.

This fact is usually stated in a different form, namely, as the problem of changing the basis of logarithms. Suppose we want to change the basis of logarithms from b to a ($a > 0$, $a \neq 1$). It can be done by using the simple formula:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

This rule is paraphrased by saying that logarithms to the bases a and b are proportional, the factor of proportionality is constant and is equal to $1/\log_b a$. It follows that if we have the logarithm table for base b, then we can easily get the one for base a. We multiply every logarithm with base b by the proportionality factor $1/\log_b a$.

Example. Given the logarithm table for base 4,

n	1	2	4	8	16	32	64	128	256	512	1024
$\log_4 n$	0	1/2	1	3/2	2	5/2	3	7/2	4	9/2	5

obtain the corresponding logarithm table for base 2.

We calculate the proportionality factor: $\frac{1}{\log_4 2} = 1/½ = 2$ and multiply all the entries in the second row of the above table to get

n	1	2	4	8	16	32	64	128	256	512	1024
$\log_2 n$	0	1	2	3	4	5	6	7	8	9	10

Note that the table for 2^n in Chapter 2 is the same as the table above for $\log_2 n$, except the two rows are interchanged.

Exercise 8: Given the logarithm table for base 10:

n	1	10	100	1000	10000	100000	1000000	10000000	100000000
$\log_{10} n$	0	1	2	3	4	5	6	7	8

find the proportionality factor $\frac{1}{\log_{10} 5}$ for base 5, given that $\log_{10} 2 = 0.301$ and fill the blanks in the table

n	1	10	100	1000						
$\log_5 n$	0	1.43								

With the help of your table calculate 100×1000; 1,000,000/10,000; 100^3; $\sqrt{1,000,000}$; $\sqrt[3]{1,000,000}$.